Geometry of Kähler metrics and holomorphic foliation by discs

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1 Introduction and Main Results

The purpose of this paper is to establish a partial regularity theory on certain homogeneous complex Monge-Ampere equations. As consequences of this new theory, we prove the uniqueness of extremal Kähler metrics and give an necessary condition for existence of extremal Kähler metrics.

Following [5], we call a Kähler metric extremal if the complex gradient of its scalar curvature is a holomorphic vector field. In particular, any Kähler metric with constant scalar curvature is extremal, conversely, if the underlying Kähler manifold has no holomorphic vector fields, then an extremal Kähler metric is of constant scalar curvature. Our first result is

Theorem 1.1. *Let* $(M, [\omega])$ *be a compact Kähler manifold with a Kähler class* $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. *Then there is at most one extremal Kähler metric with Kähler class* $[\omega]$ *modulo holomorphic transformations, that is, if* ω_1 *and* ω_2 *are extremal Kähler metrics with the same Kähler class, then there is a holomorphic transformation* σ *such that* $\sigma^* \omega_1 = \omega_2$.

The problem of uniqueness of extremal Kähler metrics has a long history. The uniqueness of Kähler-Einstein metrics was pointed out by Calabi in early 50's in the case of non-positive scalar curvature. In [2], Bando and Mabuchi proved that the uniqueness of Kähler-Einstein metric in the case of positive scalar curvature. Following a suggestion of Donaldson, the first author proved in [6] the uniqueness of Kähler metrics with constant scalar curvature in any Kähler class which admits a Kähler metric with non-positive scalar curvature. In [10], S. Donaldson proved the uniqueness of constant scalar curvature Kähler metrics with rational Kähler class on any projective manifolds (which are Kähler) without non-trivial holomorphic vector fields.[1]

The existence of extremal metrics remains open in general cases. One difficulty is due to the fact the associated equation is fully non-linear and of 4th

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[1]After we finished proving our uniqueness theorem, we learned that T. Mabuchi extended S. Donaldson's arguements to any extremal Kähler metrics with rational coefficients on any projective manifolds and proved their uniqueness in those special cases.

order. The case of Kähler-Einstein metrics has been well understood (see [20], [1], [16]).

As another consequence of our regularity theory, we will give an necessary condition on existence of Kähler metrics with constant scalar curvature in terms of the K-energy. In [12], Mabuchi introduced the following functional $\mathbf{E}_\omega$ as follows: For any φ with $\omega_\varphi = \omega + \partial\overline{\partial}\varphi > 0$, define

$$\mathbf{E}_\omega(\varphi) = -\int_0^1 \int_M \dot{\varphi}(s(\omega_{\varphi_t}) - \mu)\omega_{\varphi_t}^n \wedge dt,$$

where ω_{φ_t} is any path of Kähler metrics joining ω and ω_φ, $s(\omega_{\varphi_t})$ denotes the scalar curvature and μ is its average. It turns out that μ is determined by the first Chern class $c_1(M)$ and the Kähler class $[\omega]$.

Theorem 1.2. *Let M be a compact Kähler manifold with a constant scalar curvature Kähler metric ω. Then $\mathbf{E}_\omega(\varphi) \geq 0$ for any φ with $\omega_\varphi > 0$.*

This theorem was proved for Kähler-Einstein metrics in [2] (also see [16]) and in [6] for Kähler manifolds with non-positive first Chern class. This theorem can be also generalized to arbitrary extremal Kähler metrics by using the modified K-energy. This theorem gives a partial answer to a conjecture of the second author: M has a constant scalar curvature Kähler metric in a given Kähler class $[\omega]$ if and only if the K-energy is proper in a suitable sense on the space of Kähler metrics with the fixed Kähler class $[\omega]$. We will further discuss applications of our method here to this problem on properness in a forthcoming paper. Combining Theorem 1.2 with results in [17] and [18], we can deduce

Corollary 1.3. *Let (M, L) be a polarized algebraic manifold, that is, M is algebraic and L is a positive line bundle. If there is a constant scalar curvature Kähler metric with Kähler class equal to $c_1(L)$. Then (M, L) is asymptotically K-semistable or CM-semistable in the sense of [16] (also see [17])* [2].

The proof of these two theorems is based on studying certain homogenous Complex Monge-Ampere equations. Deep works have been done on these equations (cf. [15], [8], [6]). They are related to the geodesic equation on the space of Kähler metrics with L^2-metric.

Let M be a compact Kähler manifold with a fixed Kähler metric ω and Σ be a Riemann surface with boundary $\partial\Sigma$. Consider

$$(\pi_2^*\omega + \partial\overline{\partial}\phi)^{n+1} = 0 \ \text{ on } \Sigma \times M, \quad \phi|_{\partial\Sigma\times M} = \psi, \tag{1.1}$$

where $\pi_2 : \Sigma \times M \to M$ is the projection and ϕ is a function on $\Sigma \times M$ such that $\phi(z, \cdot) \in \mathcal{H}_\omega$ for any $z \in \Sigma$, and ψ is a given function on $\partial\Sigma \times M$ such that $\psi(z, \cdot \in \mathcal{H}$. Here $\mathcal{H}_\omega$ denotes the space of Kähler potentials

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M, \mathbb{R}) \ | \ \omega_\varphi = \omega + \partial\overline{\partial}\varphi > 0, \text{ on } M\}. \tag{1.2}$$

[2] According to [18], the CM-stability (semistability) is equivalent to the K-stability (semistability).

The proof of Theorem 1.1 and 1.2 starts with the following observations: Given two functions φ_0 and φ_1 in $\mathcal{H}_\omega$, if there is a bounded smooth solution ϕ of (1.1) on $[0,1] \times \mathbb{R} \times M$, then evaluation function $f = \mathbf{E}(\phi(z,\cdot))$ is a bounded subharmonic function which is constant along each boundary component of $[0,1] \times \mathbb{R}$, [3] furthermore, if φ_0 is a critical metric of $\mathbf{E}$, then it follows from the Maximum principle that $\mathbf{E}(\varphi_1) \geq \mathbf{E}(\varphi_0)$ and equality holds if and only if each $\phi(z,\cdot)$ is a critical metric of $\mathbf{E}$. The infinite strip $[0,1] \times \mathbb{R}$ can be approximated by discs $\Sigma_R = [0,1] \times [-R, R]$ $(R \to \infty)$. Hence, if we can show that the equation (1.1) has a uniformly bounded solution for each Σ_R[4], then Theorem 1.1 and 1.2 follow.

However, it is an extremely difficult problem to solve degenerate complex Monge-Ampere equations. Higher regularity was obtained by L. Cafferali, J. Kohn, L. Nirenberg and J. Spruck in 80's for nondegenerate complex Monge-Ampere equations under certain convexity assumptions. Weak solutions for homogenous complex Monge Ampere equations, say in L^p or $W^{1,p}$-norms, were extensively studied (cf. [3]). In [6], the first named author proved the following theorem, which plays a fundamental role in this paper.

Theorem 1.4. *([6]) For any smooth map $\psi : \partial\Sigma \to \mathcal{H}$, (1.1) always has a unique $C^{1,1}$-solution ϕ on $\Sigma \times M$ such that $\phi = \psi$ along $\partial\Sigma$.* [5] *Moreover, the $C^{1,1}$ bound of ϕ depends only on the C^2 bound of ψ.*

Our new technical contribution is to establish partial regularity of solutions from the above theorem in the case of Σ being a disc. Precisely, we prove

Theorem 1.5. *Let Σ be a holomorphic disc. For a generic boundary map $\psi : \partial\Sigma \to \mathcal{H}_\omega$, there exists a unique $C^{1,1}$ solution ϕ of (1.1) with the following properties: There is an open and dense subset $\mathcal{R}_\phi \subset \Sigma \times M$ such that*

1. *$\mathcal{R}_\phi$ is open and dense in $\Sigma \times M$, and the varying volume form $\omega^n_{\phi(z,\cdot)}$ extends to a continuous function on $\Sigma_0 \times M$, where $\Sigma_0 = (\Sigma \backslash \partial\Sigma)$. Moreover, it is positive in $\mathcal{R}_\phi$ and vanishes identically on its complement;*

2. *The distribution $\mathcal{D}_\phi$(cf. equation (1.5)) extends to a continuous distribution in a open saturated*[6] *set $\tilde{\mathcal{V}} \subset \Sigma \times M$, such that the complement S_ϕ of $\tilde{\mathcal{V}}_\phi$ is locally extendable*[7] *and ϕ is C^1 continuous on $\tilde{\mathcal{V}}$. The set S_ϕ is referred as the singular set of ϕ.*

3. *The leaf vector field $\frac{\partial}{\partial z} + v$ of $\mathcal{D}_\phi$ is uniformly bounded in $\tilde{\mathcal{V}}_\phi$.*

[3]This follows from the convexity of the K-energy $\mathbf{E}$

[4]Σ_R has four corners, but we can easily smooth corners to get a smooth Riemann surface of disc type and with boundary and use smoothed ones to approximate the given infinite strip.

[5]It is not known if $\phi(z,\cdot)$ lies in $\mathcal{H}$, but the first author proved that $\phi(z,\cdot)$ is always the limit of functions in $\mathcal{H}$.

[6]Any maximal extension of the leaf vector field lies completely inside $\tilde{\mathcal{V}}_\phi$.

[7]A closed subset $S \subset \Sigma \times M$ of measure 0 is *locally extendable* if for any continuous function in $\Sigma \times M$ which is $C^{1,1}$ on $\Sigma \times M \setminus S$ can be extended to a $C^{1,1}$ function on $\Sigma \times M$. Notice that any set of codimension 2 or higher is automatically locally extendable.

We will call the solution in the above theorem an almost smooth solution. The partial regularity in Theorem 1.5 is sharp since we do have examples where the solution for (1.1) is singular. It seems to be the first time to use singular foliations systematically to study partial regularity for homogeneous complex Monge-Ampere equations. Now let us explain briefly how this theorem is proved.

It has been known for long that solutions of homogeneous complex Monge-Ampere equations are closely related to foliations by holomorphic curves (cf. [11], [15], [9]). In [15], S. Semmes formulated the Dirichlet problem for (1.1) in terms of a foliation by holomorphic curves with boundary in a totally real submanifold in the complex cotangent bundle of the underlying manifold. Let us first recall Semmes' construction. We associate a hyperKähler manifold $\mathcal{W}_{[\omega]}$ to each Kähler class $[\omega]$: Let $\{U_i\}$ be a covering of M such that $\omega|_{U_i} = \sqrt{-1}\partial\overline{\partial}\rho_i$, we identify $(x, v_i) \in T^*U_i$ with $(y, v_j) \in T^*U_j$ if $x = y \in U_i \cap U_j$ and $v_i = v_j + \partial(\rho_i - \rho_j)$, then $\mathcal{W}_{[\omega]}$ consists of all these equivalence classes of $[x, v_i]$. There is an natural map $\pi : \mathcal{W}_{[\omega]} \mapsto T^*M$, assigning $(x, v_i) \in T^*U_i$ to $(x, v_i - \partial\rho_i)$. Then the complex structure on T^*M pulls back to a complex structure on $\mathcal{W}_{[\omega]}$ and there is also a canonical holomorphic 2-form Ω on $\mathcal{W}_{[\omega]}$, in terms of local coordinates z_α, ξ_α $(\alpha = 1, \cdots, n)$ of T^*U_i,

$$\Omega = dz_\alpha \wedge d\xi_\alpha.$$

Now for any $\varphi \in \mathcal{H}_{[\omega]}$, we can associate a complex submanifold Λ_φ in $\mathcal{W}_{[\omega]}$: For any open subset U on which ω can be written as $\sqrt{-1}\partial\overline{\partial}\rho$, we define $\Lambda_\varphi|_U =$ to be the graph of $\partial(\rho + \varphi)$. Clearly, this Λ_φ is independent of the choice of U. A straightforward computation shows

$$\Omega|_{\Lambda_\varphi} = -\sqrt{-1}\omega_\varphi, \tag{1.3}$$

that is, $\mathrm{Re}(\Omega)|_{\Lambda_\varphi} = 0$ and $-\mathrm{Im}(\Omega)|_{\Lambda_\varphi} = \omega_\varphi > 0$. This means that Λ_φ is an exact Lagrangian symplectic submanifold of $\mathcal{W}_{[\omega]}$ with respect to Ω. Conversely, given an exact Lagrangian symplectic submanifold Λ of $\mathcal{W}_{[\omega]}$, we can construct a smooth function φ such that $\Lambda = \Lambda_\varphi$. Hence, Kähler metrics with Kähler class $[\omega]$ are in one-to-one correspondence with exact Lagrangian symplectic subamnifolds in $\mathcal{W}_{[\omega]}$.

Let ψ be a smooth function on $\partial\Sigma \times M$ such that $\psi(\tau, \cdot) \in \mathcal{H}_{[\omega]}$ for any $\tau \in \partial\Sigma$. Define

$$\mathbf{\Lambda}_\psi = \{(\tau, v) \in \partial\Sigma \times \mathcal{W}_{[\omega]} \mid v \in \mathbf{\Lambda}_{\psi(\tau,\cdot)} \}. \tag{1.4}$$

One can show that $\mathbf{\Lambda}_\psi$ is a totally real submanifold in $\Sigma \times \mathcal{W}_{[\omega]}$. Now let us recall a result from [15] and [9].

Proposition 1.6. *Assume that Σ is simply connected. There is a solution ϕ of (1.1) if and only if there is a smooth family of holomorphic maps $h_x : \Sigma \mapsto \mathcal{W}_{[\omega]}$ parametrized by $x \in M$ satisfying: (1) $\pi(h_x(z_0)) = x$, where z_0 is a given point in $\Sigma\backslash\partial\Sigma$; (2) $h_x(\tau) \in \mathbf{\Lambda}_{\psi(\tau,\cdot)}$ for each $\tau \in \partial\Sigma$ and $x \in M$; (3) For each $z \in \Sigma$, the map $\gamma_z(x) = \pi(h_x(z))$ is a diffeomorphism of M.*

For the readers' convenience, let us explain briefly its proof. Let ϕ be a solution of (1.1) on $\Sigma \times M$ such that $\phi(z, \cdot) \in \mathcal{H}$ for any $z \in \Sigma$. Define $\mathcal{D}_\phi \subset T(\Sigma \times M)$ by

$$\mathcal{D}_\phi = \{ \frac{\partial}{\partial z} + v \in T_{(z,p)}(\Sigma \times M) \mid i_{\frac{\partial}{\partial z}+v}(\pi_2^*\omega + \partial\overline{\partial}\phi) = 0 \}, \quad (z,p) \in \Sigma \times M. \quad (1.5)$$

Then $\mathcal{D}$ is a holomorphic integrable distribution. If Σ is simply-connected and $\phi(z, \cdot) \in \mathcal{H}$ for each $z \in \Sigma$, then the leaf of $\mathcal{D}$ containing (z_0, x) is the graph of a holomorphic map $f_x : \Sigma \mapsto M$ with $f_x(z_0) = x$. If we write $f_x(z) = \sigma_z(x)$ we get a family of diffeomorphisms σ_z of M with $\sigma_{z_0} = \mathrm{Id}_M$. Now for any fixed z we have a Kähler form $\omega + \sqrt{-1}\partial\overline{\partial}\phi(z, \cdot)$ on M and hence a section $s_z : M \mapsto \mathcal{W}_{[\omega]}$ whose image is an exact Lagrangian symplectic graph $\Lambda_{\phi(z,\cdot)}$. Then $h_x(z) = \gamma_z(x) = s_z(f_x(z))$ as required. This process can be reversed.

In [9], S. Donaldson used this fact to study deformation of smooth solutions for (1.1) when the boundary value varies. Theorem 1.5 is proved by establishing existence of foliations by holomorphic disks with relatively mild singularity, more precisely, we will show that for a generic boundary value, there is an open set in the *moduli* space of holomorphic discs which generates a foliation on $\Sigma \times M \backslash S$ for a closed subset S of codimension at least one.

Now let us fix a generic boundary value ψ and study the corresponding *moduli* $\mathcal{M}_\psi$ of holomorphic discs. First it follows from the Index theorem that the expected dimension of this *moduli* is $2n$. Recall that a holomorphic disc u is regular if the linearized $\overline{\partial}$-operator $\overline{\partial}_u$ has vanishing cokernel. The *moduli* space is smooth near a regular holomorphic disc. Following [9], we call u super-regular if there is a basis $s_1, \cdots, s_{2n}$ of the kernel of $\overline{\partial}_u$ such that $d\pi(s_1)(x), \cdots, d\pi(s_{2n})(x)$ span $T_{u(x)}M$ for every $x \in \Sigma$, where $\pi : \mathcal{W}_{[\omega]} \mapsto M$ is the natural projection. We call u almost super-regular if $d\pi(s_1)(x), \cdots, d\pi(s_{2n})(x)$ span $T_{u(x)}M$ for every $x \in \Sigma \backslash \partial\Sigma$. Clearly, the set of super-regular discs is open.

One of our crucial observations is that Semmes' arguments can be made local along super-regular holomorphic discs.

Theorem 1.7. *For a generic boundary value ψ, an almost smooth solution of (1.1) corresponds to a nearly smooth foliation which can be described as follows: There is an open subset $\mathcal{U}_\psi \subset \mathcal{M}_\psi$ of super-regular discs such that the images of these discs in $\Sigma \times M$ give rise to a foliation on an open-dense set of $\Sigma \times M$ such that*

1. *this foliation can be extended to be a continuous foliation by holomorphic discs in an open set $\tilde{\mathcal{V}}_{\phi_0} \subset \Sigma_0 \times M$ such that it admits a continuous lifting in $\Sigma \times \mathcal{W}_M$;*

2. *the complement of $\tilde{\mathcal{V}}_{\phi_0}$ in $\Sigma_0 \times M$ is locally extendable.*

3. *The leaf vector (cf. equation 1.5) induced by the foliation in $\mathcal{V}_{\phi_0}$ is uniformly bounded.*

We can prove that these locally constructed solution are actually compatible to each other if their domains overlap. If the set of super regular discs is M,

then these locally constructed solutions give rise to a smooth solution to (1.1). In general, one gets only a solution to (1.1) in $(\Sigma \times M) \setminus S$ with appropriate boundary condition on $(\partial\Sigma \times M) \setminus S$. We can apply the Maximum Principle along super-regular leaves to get a uniform $C^{1,1}$-bound ϕ on $(\Sigma \times M) \setminus S$. This uniform $C^{1,1}$-bound can be used to get a solution by patching together local solutions to (1.1) along super-regular leaves.

Theorem 1.5 will follow from the following

Theorem 1.8. *For a generic boundary value ψ, there is a nearly smooth foliation generated by an open set of the corresponding moduli space $\mathcal{M}_\psi$. Definition of nearly smooth foliation is already given in the statement of Theorem 1.7.*

Now we outline the proof of Theorem 1.8. Let ψ be a generic boundary value such that $\mathcal{M}_\psi$ is smooth. This follows from a result of Oh [13] on transversality. By a similar (but different) transversality argument, one can show that there is a generic path ψ_t $(0 \le t \le 1)$ such that $\psi_0 = 0$ and $\psi_1 = \psi$ and the total *moduli* space $\tilde{\mathcal{M}} = \bigcup_{t\in[0,1]} \mathcal{M}_{\psi_t}$ is smooth, moreover, we may assume that $\mathcal{M}_{\psi_t}$ are smooth for all t except finitely many $t_1, \cdots, t_N$ where the moduli space may have isolated singularities. It follows from Semmes and Donaldson (1.6) that $\mathcal{M}_0$ has at least one component which gives a foliation for $\Sigma \times M$. We want to show that this component will deform to a component of $\mathcal{M}_\psi$ which generates a nearly smooth foliation. We will use the continuity method. Assume that ϕ is the unique $C^{1,1}$-solution of (1.1) with boundary value ψ_t for some $t \in [0, 1]$. Let f be any holomorphic disc in the component of $\mathcal{M}_{\psi_t}$ which generates the corresponding foliation.

Lemma 1.9. *There is a uniform upper bound on area of f, where the area is with respect to the induced metric on $\mathcal{W}_{[\omega]}$ by $dzd\bar{z}$ on Σ and ω on M.*

Proof. First we observe

$$\mathrm{Area} f(\Sigma) \le C\sqrt{-1}\int_\Sigma f^*(dz \wedge d\bar{z} + \omega), \tag{1.6}$$

where C is a constant depending only on $C^{1,1}$-norm of ϕ. By direct computations using (1.1), we have $\partial\overline{\partial}\phi(z, f(z)) = -f^*\omega$. Then, integrating by parts, we can bound the area of f in terms of the area of Σ and the $C^{1,1}$-bound of ϕ. □

By Gromov's compactness theorem, any sequence of holomorphic discs with uniformly bounded area has a subsequence which converges to a holomorphic disc together with finitely many bubbles. These bubbles which occur in the interior are holomorphic spheres, while bubbles in the boundary might be holomorphic spheres or discs. We will show that no bubbles can actually occur.

For a fixed totally real submanifold, holomorphic bubbles can not occur in the boundary since boundaries of discs lie in a fixed totally real submanifold. If a sequence of totally real submanifolds converges to a given totally real submanifold, there are two limiting processes, one concerns how fast the bubbles form and move to the boundaries of discs, while the other is about how fast the

sequence of totally real submanifolds approaches to the limiting submanifold. The uniform $C^{1,1}$ bound on ϕ can be used to show that the two limiting processes are exchangeable. Consequently, one can show that there are no bubbles along boundary.

We can also rule out bubbles in the interior of discs. Heuristically speaking, an interior bubble corresponds to a holomorphic map from S^2 into the target manifold $\mathcal{H}$. According to E. Calabi and X. Chen [7], this infinite dimensional space $\mathcal{H}$ is non-positively curved in the sense of Alexanderov, consequently, there are no non-trivial holomorphic spheres in $\mathcal{H}_{[\omega]}$! This heuristical argument implies that there are no interior bubbles. Indeed, there is a rigorous proof for this fact. The proof is much more involved and will be presented elsewhere.

Since there are no bubbles arised either in the boundary or interior of the disc Σ, the Fredholm index of holomorphic discs is invariant in limiting process. This is an important fact needed in our doing deformation theory.

In order to get a nearly smooth foliation, we need to prove that the *moduli* space has an open set of super-regular holomorphic discs for each t. First we observe that the set of super-regular discs is open. Moreover, using the transversality arguments, one can show that for a generic path ψ_t, the closure of all super-regular discs in each $\mathcal{M}_{\psi_t}$ is either empty or forms an irreducible component. It implies the openness. It remains to proving that each moduli has at least one super-regular disc. It is done by using capacity estimate which we explain briefly in the following.

Consider the bundle $\mathcal{E} = \pi_2^* TM$ over $\Sigma \times M$. Each almost smooth solution ϕ of (1.1) induces an Hermitian metric on $\mathcal{E}|_{\mathcal{R}_\phi}$, where $\mathcal{R}_\phi$ was defined in Theorem 1.5. If f is a super-regular disc, then $\mathcal{E}$ pulls back to an Hermitian bundle over Σ with fiber $T_{f(z)}M$ and metric $\omega_{\phi(z,\cdot)}(f(z))$ over $z \in \Sigma$. It turns out that the curvature of this Hermitian bundle is non-positive. This fact plays a crucial role in our work. More precisely, we have

Lemma 1.10. *Let ϕ be a solution of (1.1) and f be a super-regular holomorphic disc as above, then the curvature form F of the metric g_ϕ described above is given by*

$$g_\phi(F(u), v) = -g_\phi(u(\overline{\partial_z f}), v(\overline{\partial_z f})) \qquad u, v \in TM.$$

In particular, the curvature is non-positive. Moreover, the foliation is holomorphic along f if and only if the curvature vanishes.

The determinant $\wedge^n \mathcal{E}$ restricts to an Hermitian line bundle over any given super-regular disc. The corresponding Hermitian metric, denoted by $f^*\omega_\phi^n$, at z is $\omega^n_{\phi(z,\cdot)}(f(z))$. An immediate corollary of above lemma is that the curvature of this line bundle is non-positive [8]. Moreover, there are constants C_1, C_2 which depend only on the background metric ω such that

$$\Delta \left(\log \frac{f^*\omega_\phi^n}{f^*\omega^n} + C_1 \varphi \right) \geq 0, \tag{1.7}$$

[8]This fact was first proved by S. K. Donaldson and X. X. Chen independently (via different methods) while both of them were visiting at Stanford University.

and

$$\Delta\left(\log\frac{f^*\omega_\varphi^n}{f^*\omega^n}+C_2\varphi\right)\le -\mathrm{tr}(F), \tag{1.8}$$

where Δ denotes the standard Laplacian operator on Σ and $\varphi(z)=\phi(z,f(z))$. It follows that $\log\frac{f^*\omega_\phi^n}{f^*\omega^n}+C_1\varphi$ is subharmonic and uniformly bounded on the boundary $\partial\Sigma$. The $C^{1,1}$-estimate in [6] implies that this function is uniformly bounded from above. Moreover, the difference of two functions $\log\frac{f^*\omega_\phi^n}{f^*\omega^n}+C_1\varphi$ and $\log\frac{f^*\omega_\phi^n}{f^*\omega^n}+C_2\varphi$ is uniformly bounded. In addition, we have

$$-\Delta\mathrm{tr}(F)\ge\frac{2}{n}(-\mathrm{tr}(F))^2 \tag{1.9}$$

for some positive constant c. Following [14] and [4], one can use this differential inequality to derive an interior estimate on $\mathrm{tr}(F)$ (details will appear elsewhere). Applying this estimate on $\mathrm{tr}(F)$ to the above equations, we can derive an Harnack-type inequality $\frac{f^*\omega_\phi^n}{f^*\omega^n}$ in the interior of Σ.

Now let us introduce the notion of Capacity for super-regular holomorphic discs:

Definition 1.11. *For any super-regular disc f in an moduli space $\mathcal{M}_\psi$, we define its capacity by*

$$Cap(f)=\frac{\sqrt{-1}}{2}\int_\Sigma\frac{f^*\omega^n}{f^*\omega_\phi^n}\,dz\wedge d\bar z.$$

Using the Harnack-type inequality mentioned above, one can control the lower bound of $\frac{f^*\omega_\phi^n}{f^*\omega^n}$ in the interior of σ in terms of upper bound of the capacity of f. This has an important corollary for compactness of super-regular discs with uniformly bounded capacity.

Theorem 1.12. *Let f_i be any sequence of super-regular discs in $\mathcal{M}_{\psi_{t_i}}$ which converges smoothly to an embedded disc f_∞ in $\mathcal{M}_{\psi_{t_\infty}}$. If the capacities $Cap(f_i)$ are uniformly bounded, then the limiting disc f_∞ is also super-regular.*

In fact, Lemma 1.10 was already needed when we extended Semmes' correspondence to almost smooth solutions of (1.1) and nearly smooth foliations. For this local extension, we first construct smooth solutions of (1.1) along super-regular leaves and then glue them together to a solution ϕ on an open and dense subset $V_{good}\subset\Sigma\times M$, but we need to establish $C^{1,1}$-bound of ϕ. Once this bound is established, the maximum principle implies that ϕ coincides with the solution in [6]. The $C^{1,1}$-bound of ϕ follows from the following

Theorem 1.13. *For any global holomorphic section $s:\Sigma\to\mathcal{E}$, the norm of s with respect to g_ϕ achieves its maximum value at the boundary of the disc.*

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